SUBGROUP-BASED RANK-1 LATTICE QUASI-MONTE CARLO

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1. INTRODUCTION

Integration Problem:

$$I(f) := \int_{[0,1]^d} f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

Equal-weight Approximation:

$$Q(f) := \frac{1}{n}\sum_{i=1}^{n} f(\boldsymbol{x}_i)$$

Our Contributions:

- We propose a simple and efficient closed-form method for rank-1 lattice construction, which does not require the time-consuming exhaustive computer search that previous rank-1 lattice algorithms rely on.
- A side product is a closed-form method to generate QMC points set on sphere $\mathbb{S}^{d-1}$ with bounded mutual coherence.

2. CONSTRUCTION OF OUR RANK-1 LATTICE

Construction Formula:

$$\boldsymbol{x}_i := \frac{i\boldsymbol{z} \bmod n}{n}, i \in \{0, ..., n-1\}.$$

where $\boldsymbol{z} \in \mathbb{Z}^d$ is the generating vector.

Closed-form Generating vector:

$$\boldsymbol{z} = [g^0, g^{\frac{n-1}{2d}}, g^{\frac{2(n-1)}{2d}}, \cdots, g^{\frac{(d-1)(n-1)}{2d}}] \bmod n$$

where g is a primitive root modulo a prime number n.

3. PROPERTIES OF OUR RANK-1 LATTICE

Property 1: Regular Distance Pattern

Theorem 1 *Suppose n is a prime number and $2d|(n-1)$. Let g be a primitive root of n. Let $\boldsymbol{z} = [g^0, g^{\frac{n-1}{2d}}, g^{\frac{2(n-1)}{2d}}, \cdots, g^{\frac{(d-1)(n-1)}{2d}}]$ mod n. Construct a rank-1 lattice $X = \{\boldsymbol{x}_0, \cdots, \boldsymbol{x}_{n-1}\}$ with $\boldsymbol{x}_i = \frac{i\boldsymbol{z} \bmod n}{n}, i \in \{0, ..., n-1\}$. Then, there are $\frac{n-1}{2d}$ distinct pairwise toroidal distance values among X, and each distance value is taken by the same number of pairs in X.*

Remark: Our subgroup-based rank-1 lattice has a more regular pattern with fewer distinct pairwise distance values.

Property 2: Bounded Toroidal Distance

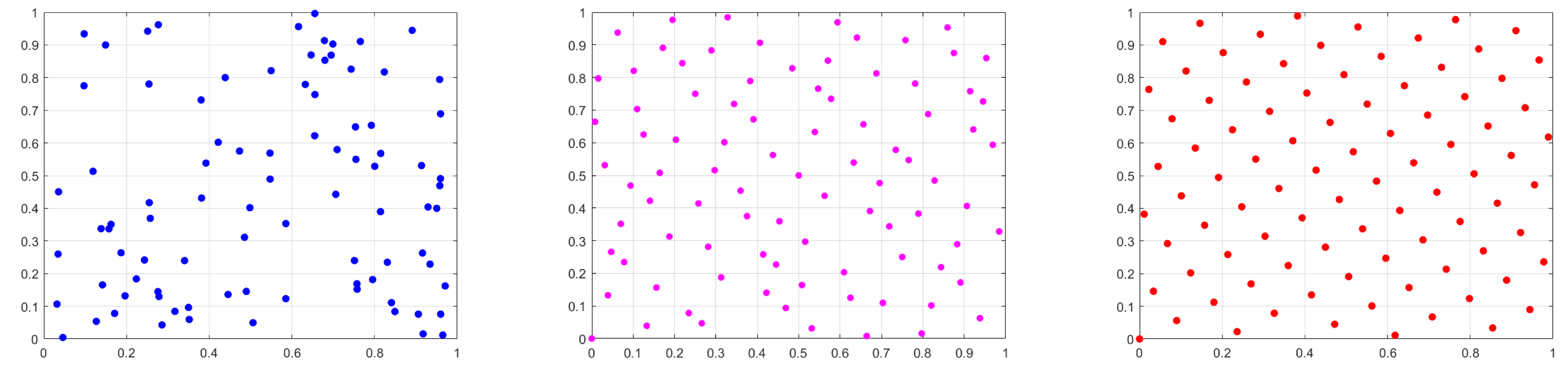
Theorem 2 *Suppose n is a prime number and $n \geq 2d + 1$. Let $\boldsymbol{z} = [z_1, z_2, \cdots, z_d]$ with $1 \leq z_k \leq n-1$. Construct a rank-1 lattice $X = \{\boldsymbol{x}_0, \cdots, \boldsymbol{x}_{n-1}\}$ with $\boldsymbol{x}_i = \frac{i\boldsymbol{z} \bmod n}{n}, i \in \{0, ..., n-1\}$ and $z_i \neq z_j$. Then, the minimum pairwise toroidal distance can be bounded as*

$$\frac{d(d+1)}{2n} \leq \min_{i,j\in\{0,\cdots,n-1\}, i\neq j} \|\mathbf{x}_i - \mathbf{x}_j\|_{T_1} \leq \frac{(n+1)d}{4n}$$

$$\frac{\sqrt{6d(d+1)(2d+1)}}{6n} \leq \min_{i,j\in\{0,\cdots,n-1\}, i\neq j} \|\mathbf{x}_i - \mathbf{x}_j\|_{T_2} \leq \sqrt{\frac{(n+1)d}{12n}},$$

where $\|\cdot\|_{T_1}$ and $\|\cdot\|_{T_2}$ denote the l_1-norm-based toroidal distance and the l_2-norm-based toroidal distance, respectively.

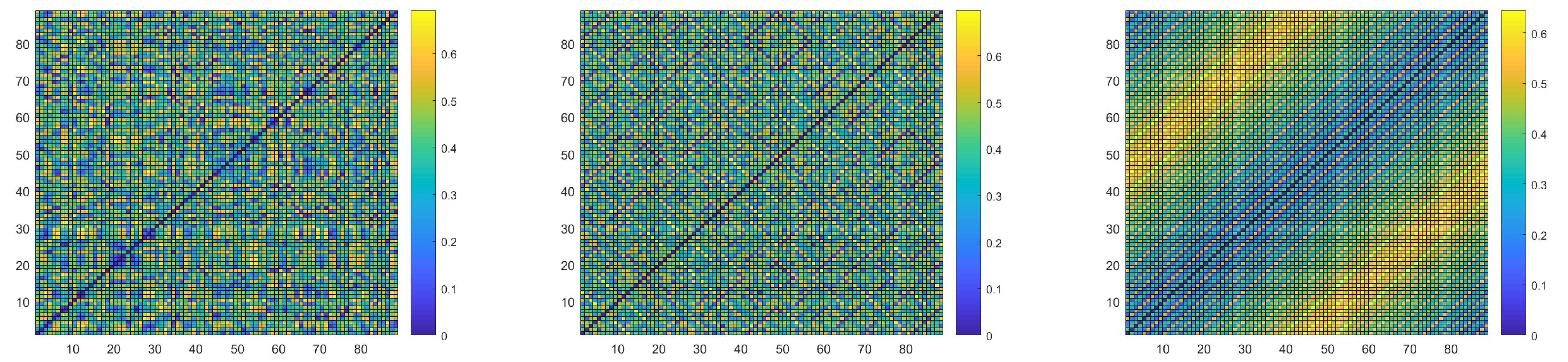
4. AN ILLUSTRATION OF SUBGROUP-BASED RANK-1 LATTICE



(a) i.i.d. Monte Carlo sampling (b) Sobol sequence (c) Our subgroup rank-1 lattice

Figure 1: The 89 points constructed by i.i.d. Monte Carlo sampling, Sobol sequence and our subgroup rank-1 lattice on $[0, 1]^2$.

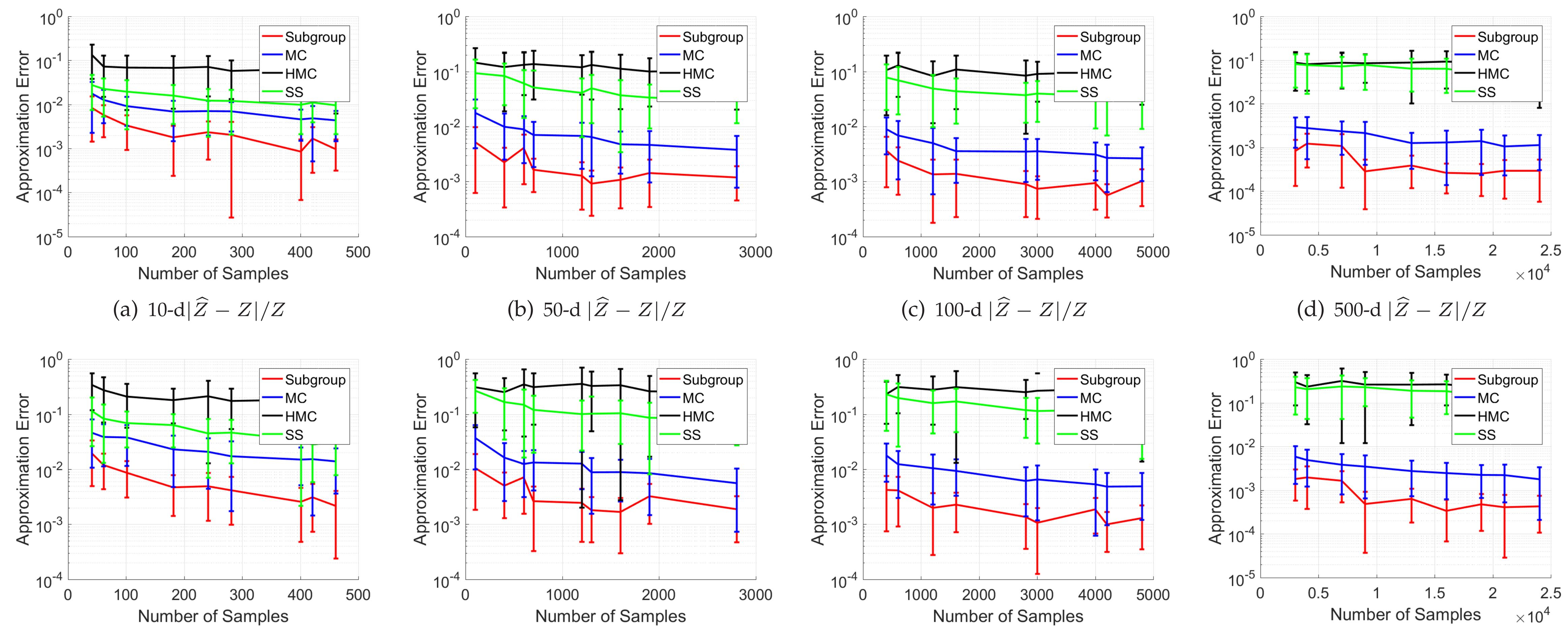
5. REGULAR DISTANCE PATTERN



(a) i.i.d. Monte Carlo sampling (b) Sobol sequence (c) Our subgroup rank-1 lattice

Figure 2: Pairwise distance constructed by i.i.d. Monte Carlo sampling, Sobol sequence and our subgroup rank-1 lattice on $[0, 1]^2$.

6. APPROXIMATION OF THE NORMALIZATION CONSTANT AND MARGINAL LIKELIHOOD



(a) 10-d $|\widehat{Z} - Z|/Z$ (b) 50-d $|\widehat{Z} - Z|/Z$ (c) 100-d $|\widehat{Z} - Z|/Z$ (d) 500-d $|\widehat{Z} - Z|/Z$

(e) 10-d $|\widehat{\mathcal{L}} - \mathcal{L}|/\mathcal{L}$ (f) 50-d $|\widehat{\mathcal{L}} - \mathcal{L}|/\mathcal{L}$ (g) 100-d $|\widehat{\mathcal{L}} - \mathcal{L}|/\mathcal{L}$ (h) 500-d $|\widehat{\mathcal{L}} - \mathcal{L}|/\mathcal{L}$

Figure 3: Mean approximation error over 50 independent runs. Error bars are with in $1\times$ std

7. EXPERIMENTAL RESULTS

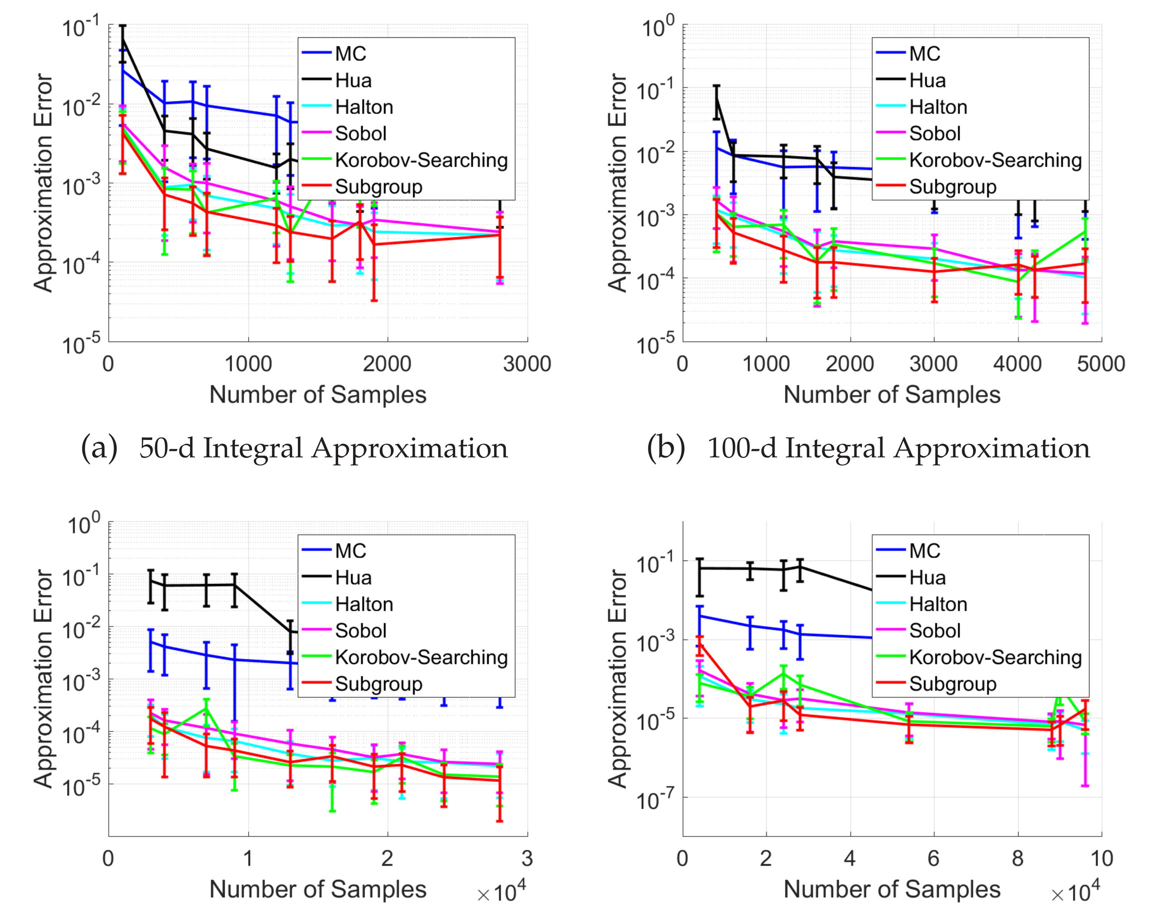
Experiment 1: Comparison of the minimum l_2-norm-based toroidal distance of rank-1 lattice constructed by different methods.

d=50		n=101	401	601	701	1201	1301	1601	1801	1901	2801
	SubGroup	**2.0513**	**1.9075**	**1.9469**	**1.9196**	**1.8754**	1.8019	1.8008	**1.8709**	1.7844	1.7603
	Hua	1.7862	1.7512	1.7293	1.7049	1.7326	1.6295	1.6659	1.6040	1.5629	1.5990
	Korobov	**2.0513**	**1.9075**	**1.9469**	**1.9196**	**1.8754**	**1.8390**	**1.8356**	**1.8709**	**1.8171**	**1.8327**
d=100		401	601	1201	1601	1801	2801	3001	4001	4201	4801
	SubGroup	**2.8342**	**2.8143**	2.7077	**2.7645**	**2.7514**	2.6497	2.6337	2.6410	2.6195	2.5678
	Hua	2.5385	2.5739	2.4965	2.4783	2.4132	2.5019	2.4720	2.4138	2.4537	2.4937
	Korobov	**2.8342**	**2.8143**	**2.7409**	**2.7645**	**2.7514**	**2.6956**	**2.6709**	**2.6562**	**2.6667**	**2.6858**
d=200		401	1201	1601	2801	4001	4801	9601	12401	14401	15601
	SubGroup	**4.0876**	**3.9717**	**3.9791**	3.8425	**3.9276**	3.8035	3.7822	**3.8687**	3.6952	3.8370
	Hua	3.7332	3.7025	3.6902	3.6944	3.7148	3.6936	3.6571	3.5625	3.6259	3.5996
	Korobov	**4.0876**	**3.9717**	**3.9791**	**3.9281**	**3.9276**	**3.9074**	**3.8561**	**3.8687**	**3.8388**	**3.8405**
d=500		3001	4001	7001	9001	13001	16001	19001	21001	24001	28001
	SubGroup	**6.3359**	**6.3769**	6.3141	6.2131	**6.2848**	6.2535	6.0656	**6.2386**	**6.2673**	6.1632
	Hua	5.9216	5.9216	5.9215	5.9215	5.9216	5.9216	5.9215	5.9215	5.8853	5.9038
	Korobov	**6.3359**	**6.3769**	**6.3146**	**6.2960**	**6.2848**	**6.2549**	**6.2611**	**6.2386**	**6.2673**	**6.2422**

Experiment 2: Time Comparison of Korobov searching and our sub-group rank-1 lattice.

d=500		n=3001	4001	7001	9001	13001	16001	19001	21001	24001	28001
	SubGroup	0.0185	0.0140	0.0289	0.043	0.0386	0.0320	0.0431	0.0548	0.0562	0.0593
	Korobov	34.668	98.876	152.86	310.13	624.56	933.54	1308.9	1588.5	2058.5	2815.9
d=1000		n=4001	16001	24001	28001	54001	70001	76001	88001	90001	96001
	SubGroup	0.0388	0.0618	0.1041	0.1289	0.2158	0.2923	0.3521	0.4099	0.5352	0.5663
	Korobov	112.18	1849.4	4115.9	5754.6	20257	34842	43457	56798	56644	69323

Experiment 3: Comparison of our subgroup-based rank-1 lattice with other baselines on integral approximation problems.



(a) 50-d Integral Approximation (b) 100-d Integral Approximation

(c) 500-d Integral Approximation (d) 1000-d Integral Approximation

Figure 4: Mean approximation error over 50 independent runs.error bars are with in $1\times$ std.

8. CONCLUSION

- We proposed a closed-form subgroup-based rank-1 lattice for integral approximation without computer searching. Our subgroup rank-1 lattice has few different pairwise distance values, which is more regular to be evenly spaced.
- In addition, we propose a closed-form method to generate QMC points set on sphere $\mathbb{S}^{d-1}$. We proved upper bounds of the mutual coherence of the generated points.
- Our subgroup-based rank-1 lattice and QMC on sphere $\mathbb{S}^{d-1}$ can be used for Bayesian inference, kernel approximation, generative models training, and the approximation of Wasserstein distance. It may also be able to combine with sequential adaptive MC to improve performance.